Again about Andrica's Conjecture...

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Abstract: The paper establishes an equivalence of the Andrica's conjecture in the direction of an increase of the difference of square root of primes by a combination of two consecutive primes.

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1. Introduction

In a previous paper, entitled "About Andrica's conjecture" the authors have established an equivalence of conjecture Andrica by considering the ratio of two consecutive prime numbers. Because the average deviation calculated relative to the two terms, in this article will study another limit for the difference of square roots of two consecutive prime numbers.

A number $p\in\mathbf{N}$, $p\geq 2$ is called prime number if its only positive divisors are 1 and p.

Even if do not know much about prime numbers, there exist a lot of attempts to determine some of their properties, many results being at the stage of conjectures.

A famous conjecture relative to prime numbers is that of Dorin Andrica. Denoting by p_n - the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$ etc.), Andrica's conjecture ([1]) states that:

$$\sqrt{p_{n+1}}-\sqrt{p_n}<1 \quad \forall n\geq 1$$

In [3] we have found the following:

Theorem

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Let p_n the n-th prime number. The following statements are equivalent for n≥5:

1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$;
2. ∃α≥0 such that: $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{p_n}{p_{n+1}}\right)^{\alpha}$.

In the following, we shall prove a stronger theorem of equivalence of Andrica's conjecture.

2. Main Theorem

Theorem

Let p_n the n-th prime number. The following statements are equivalent for n≥5:

1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$;
2. ∃α≥0 such that: $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{p_{n+1}^{\,p_n}}{p_n^{\,p_{n+1}}}\right)^{\alpha}$.

Proof

First of all let the function f:[e,∞)→**R**, $f(x) = \frac{\ln x}{x}$. We have: $f'(x) = \frac{1-\ln x}{x^2}<0$ therefore f is a strictly decreasing function. For n≥2 we have therefore: $f(p_n) > f(p_{n+1})$ that is: $\frac{\ln p_n}{p_n} > \frac{\ln p_{n+1}}{p_{n+1}} \Leftrightarrow p_{n+1}\ln p_n > p_n \ln p_{n+1} \Leftrightarrow p_{n+1}^{\,p_n} < p_n^{\,p_{n+1}}$ $\Leftrightarrow \frac{p_{n+1}^{\,p_n}}{p_n^{\,p_{n+1}}}<1$.

<u>2⇒1</u> Because $\frac{p_{n+1}^{\,p_n}}{p_n^{\,p_{n+1}}}<1$ follows that: α≥0 ⇒ $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{p_{n+1}^{\,p_n}}{p_n^{\,p_{n+1}}}\right)^{\alpha} < \left(\frac{p_{n+1}^{\,p_n}}{p_n^{\,p_{n+1}}}\right)^{0}=1$.

<u>1⇒2</u> If we take the logarithm in the relationship, it becomes:

$$\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\ln\left(\frac{p_{n+1}{}^{p_n}}{p_n{}^{p_{n+1}}}\right)\Leftrightarrow\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\left(\ln p_{n+1}{}^{p_n}-\ln p_n{}^{p_{n+1}}\right)\Leftrightarrow$$

$$\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\left(p_n\ln p_{n+1}-p_{n+1}\ln p_n\right)\Leftrightarrow\frac{\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)}{p_{n+1}\ln p_n-p_n\ln p_{n+1}}<-\alpha\Leftrightarrow$$

$$\frac{\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)}{\sqrt{p_{n+1}}^2\ln\sqrt{p_n}-\sqrt{p_n}^2\ln\sqrt{p_{n+1}}}<-2\alpha.$$

Let now the function:

$$g:(a,\infty)\to\mathbf{R},\ g(x)=\frac{\ln(x-a)}{x^2\ln a-a^2\ln x}\ \text{with } a>2$$

We have now: $g'(x)=\dfrac{\frac{1}{x-a}\left(x^2\ln a-a^2\ln x\right)-\ln(x-a)\left(2x\ln a-\frac{a^2}{x}\right)}{\left(x^2\ln a-a^2\ln x\right)^2}=$

$$\frac{x^3\ln a-a^2x\ln x-2x^2(x-a)\ln a\ln(x-a)+a^2(x-a)\ln(x-a)}{x(x-a)\left(x^2\ln a-a^2\ln x\right)^2}$$

Because the denominator of g’ is positive, we must inquire into the character of the function:

$h:(a,\infty)\to\mathbf{R},\ h(x)=x^3\ln a-a^2x\ln x-2x^2(x-a)\ln a\ln(x-a)+a^2(x-a)\ln(x-a).$

Computing the derivative of h:

$$h'(x)=x^2\ln a-a^2\ln x-6x^2\ln a\ln(x-a)+4ax\ln a\ln(x-a)+a^2\ln(x-a)$$

Let now:

$y:(a,\infty)\to\mathbf{R}$,
$y(x)=x^2\ln a-a^2\ln x-6x^2\ln a\ln(x-a)+4ax\ln a\ln(x-a)+a^2\ln(x-a)$

and the derivative:

$$y'(x)=\frac{a^3-2x\left(2(x-a)(3x-a)\ln(x-a)+x(2x-a)\right)\ln a}{x(x-a)}$$

Let now the function (the numerator of y):

$$z(x)=2x\left(2(x-a)(3x-a)\ln(x-a)+x(2x-a)\right)\ln a$$

and, also, the derivative:

$$z'(x) = 4\ln a\left(\left(9x^2 - 8ax + a^2\right)\ln(x-a) + 2x(3x-a)\right)$$

Because x>a we have that $9x^2 - 8ax + a^2$>0 therefore z is a strictly increasing function.

But $\lim\limits_{x\to a} z(x) = 2a^3 \ln a$ >0 therefore $z(x)$>0 $\forall$x>a.

In this case $y'(x)$>0 then y is also a strictly increasing function.

But $y(a+1) = (a+1)^2 \ln a - a^2 \ln(a+1) = a^2(a+1)^2\left(\frac{\ln a}{a^2} - \frac{\ln(a+1)}{(a+1)^2}\right)$.

The function $u(x) = \frac{\ln x}{x^2}$ has $u'(x) = \frac{x(1-2\ln x)}{x^4}$<0 for x>2 therefore u is decreasing and $\frac{\ln a}{a^2} - \frac{\ln(a+1)}{(a+1)^2} = u(a) - u(a+1)$>0.

We have now: $y(a+1)$>0 therefore $y(x)$>0 for x≥a+1.

Now $h'(x)$>0 which give us: h is increasing.

But $h(a+1) = a^2(a+1)^3\left(\frac{\ln a}{a^2} - \frac{\ln(a+1)}{(a+1)^2}\right)$>0 therefore $h(x)$>0 for x≥a+1.

Because now: $g'(x)$>0 implies that g is increasing and with $g(a+1) = 0$ we find that g(x)>0 $\forall$x≥a+1.

From hypothesis 1 (Andrica's conjecture), we have: $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ and noting: x= $\sqrt{p_{n+1}}$, a= $\sqrt{p_n}$ we have: x∈(a,a+1) therefore: $g(a) < g(x) < g(a+1)$ $\Leftrightarrow$ $-\infty < \frac{\ln(x-a)}{x^2 \ln a - a^2 \ln x} < 0$

Considering $\alpha = -\frac{1}{2}\sup\limits_{n} \frac{\ln\left(\sqrt{p_{n+1}} - \sqrt{p_n}\right)}{\sqrt{p_{n+1}}^2 \ln\sqrt{p_n} - \sqrt{p_n}^2 \ln\sqrt{p_{n+1}}}$ ≥0, the statement 2 is now obvious. **Q.E.D.**

3. Determination of the Constant α

Using the Wolfram Mathematica software, in order to determine the constant α (for the first 100000 prime numbers):

```
Clear[“Global`*”];
numberiterations=100000;
minimum=1000;
k=0;
For[i=5,i<=numberiterations+4, i++,
difference=Sqrt[Prime[i+1]]-Sqrt[Prime[i]];
ratio=Log[Prime[i+1]]*Prime[i]-Log[Prime[i]]*Prime[i+1];
log=Log[difference]/ratio;
If[log<minimum,minimum=log]];
Print["Minimum=",N[minimum,1000]]
```

we found that the first 1000 decimals are:

α=0.0018017879091801840905588819908795818525878151886260608296711819955181532280561858686616697228936379299051501383617413579875982175209124929580001342711022482912914401002119213829596110309623520462131231077387005390210757483715140857559245718080716050728272841276437791095986635315223741002438617978237774820283643801709366814693751891246115950387010547408998353108508584812645551656342521991606233800732728344510802196819799319182876091294860973601769699925486766297165720675277209011231194017976273680037341348819649636432410477964856548589141871037205705104001937233000378597273514799515653066274683520758840998066176214741755894232208444695273825009145486710866352855099595409905960655726754630444411516619929414751645003809279755608307505023674585289341679219255473742649151215747071146227738653355338521589343137819094071193983888180282339460737562287988046049746231538931008572428480523706827673078186615016687046047567231467115620223532660819705788585430650455499896978391967058243502265073317 62.

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